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Gravity Model Performance in Inertial Navigation

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Traditional inertial navigation system error covariance analyses have modeled the gravity model error as a time domain Markov process. In general, this is not strictly valid, and is noticeably inadequate when used for closed-course trajectories. An exact error covariance analysis technique has been developed that applies for all trajectories. The method is based on a double integral expression of the navigation system error covariance. A numerical algorithm based on a pair of nested, single integrals (Nested Integrals) was developed, and was verified against linear state-space covariance analysis on a great circle case. Errors associated with the traditional linear state-space analysis were demonstrated on a minor circle case. A comparison was made to a previously published Monte Carlo analysis. The Nested Integrals method was shown to be more efficient than Monte Carlo and more flexible than linear state-space covariance analysis.

I. Introduction

INERTIAL navigation has evolved to the point that the traditional gravity model is a principle error source in advanced, precise systems. Future autonomous strategic systems, such as the cruise missile, will likely require a more accurate gravity model. Mission success depends on navigation system accuracy, not gravity model accuracy per se. So an improved gravity model should be judged by system-level performance expressed, for example, as circular-error-probable (CEP). Statistical analyses based on Monte Carlo and linear state-space techniques have been used to produce this gravity model performance index. A brief review of these methods will clarify the need for a new analysis approach.

The gravity model in an inertial navigation system (INS) is a subsystem or component providing one part of the information required for constructing position, velocity and attitude estimates. The total acceleration is given by

$$a = f + G \tag{1}$$

where f is the specific force vector, and G is the gravitational acceleration vector.

A navigation system estimate of acceleration is the sum of specific force measured by accelerometers, and of gravitational acceleration calculated using a mathematical model. The modeling errors are

$$\delta g(r) = G_m(r) - G(r) \tag{2}$$

where δg is gravity disturbance, $G_m(\cdot)$ is the gravitation model, $G(\cdot)$ is true gravitation, and r is position. The gravity disturbance of Eq. (2) introduced into Eq. (1) causes the navigational errors which are the subject of concern.

The modeling error depends on the particular model. The "ellipsoidal" gravity model traditionally has been used, and is still sufficient for all but the most exacting military and space applications. Improvements to this simple model depend on modeling method, navigation system computational resources, and data base limitations. For modeling method, several approaches exist for obtaining improvements.[1] With computational resource and data base limitations, two basic questions arise in selecting a model improvement: 1) Which mathematical form is most desirable? and 2) What level of detail (e.g., number of terms) are required to meet a specification? A method for answering such questions is required for a rational selection of a gravitation model. The performance of the gravitation model should be judged by the resultant system-level navigational uncertainties not by the model errors per se. Statistical measures such as CEP are used in the model performance assessment. So the analysis method should yield, as a minimum, second-order statistics or covariances of navigation system errors.

The general strategy of statistical analysis is shown in Fig. 1. The random process is considered to be the gravity disturbance field summarized in a statistical model. The propagation of the errors within the INS is modeled, and includes the mechanism for coupling to the gravity model errors. The trajectory model includes all environmental factors, which depend on the position-time history of the mission. Finally, an analysis method employs these three models to produce the desired system-level error statistics induced by the gravity model errors.

The three models required for overall analysis are available, or can be formed, by adapting existing models. The INS model can be a straightforward, whole-valued, double integration of Eq. (1), or it can be a well-respected first-order error propagation model.[2,3] The trajectory dynamics can be modeled by kinematic models such as PROFGEN[4] or by closed form mathematical expressions for simple trajectories, such as a great circle flight. Gravity disturbance statistics can be summarized in the form of a scalar autocorrelation function with other necessary auto- and cross-correlations derived using field theory interrelationships. Although the gravity disturbance modeling issues are complex,[5] a variety of statistical models exist: linear state space, based on a Gauss-Markov process[6]; anomaly degree variance, based on a spherical harmonic representation of an isotropic correlation function[7]; and attenuated white noise process for anomalous potential.[8]

Since modeling techniques exist for all of the three areas, attention can be concentrated on the analysis method. Previous statistical analyses have been either Monte Carlo analysis[9] or linear state space covariance analysis.[10] Limitations and costs associated with these existing methods provide the motivation for developing an alternative analysis method.

Monte Carlo Method

First, the Monte Carlo method takes a direct mission simulation approach. It is, in fact, an ensemble of deter-

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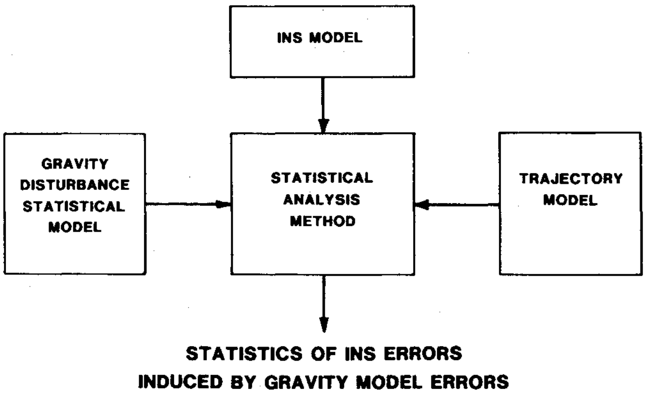

Fig. 1 General strategy of statical analysis.

ministic cases corresponding to an ensemble of residual field realizations. The gravity disturbance profiles are produced in such a way that in the limit they match the correlations given by the residual field correlation function. These simulations may be either of the navigation equations directly (whole-valued simulation), or of navigation error propagation. In either case, the ultimate output of these simulations is an ensemble of error-time histories from which means, standard deviations, covariances, and other sample statistics can be calculated.

Computation time is a significant cost of the Monte Carlo method. Enough cases must be run to yield statistics of high confidence. In a recent advanced cruise missile study, Chatfield[9] used 90 simulations to produce the desired circular-error-probable statistic.

While this cost is considerable, analysis on this complex trajectory demonstrates the flexibility of this method. The Monte Carlo restrictions on trajectory are minor, being essentially limited to the ones imposed by the trajectory simulator. In addition to this scenario flexibility, considerable freedom exists in the functional form of the correlation model. The gravity disturbance profiles are produced using the residual field correlation function. While the residual field is dependent on the gravity model, there are, as noted above, several developed functional models to summarize the correlations of the residual field. The Monte Carlo method imposes no limitation on the functional form of the correlation model.

Linear State Space Covariance Analysis

The linear state space covariance approach was formulated by Levine and Gelb in a landmark paper in 1968.[10] This method and the attendant stochastic gravity model have been refined to be consistent with most of the field theory impositions.[11]

At the heart of the linear state-space method is a first-order, linear differential equation as a model of INS error propagation:

$$\dot{x}(t) = F(t)x(t) + G(t)u(t) \tag{3}$$

where $x(t)$ is an n-element column vector of navigational errors, $F(t)$ is an $n \times n$ error propagation matrix, $u(t)$ is an m-element column vector of gavity disturbance terms, $G(t)$ is an $n \times m$ distribution matrix, and $\dot{x}(t) = \mathrm{d}x(t)/\mathrm{d}t$.

If $u(t)$ is assumed to be the output of a linear system driven by white, Gaussian noise, an overall linear covariance analysis can be performed. The underlying state vector model of this linear filter is spatial in nature:

$$\mathrm{d}x_g(\psi)/\mathrm{d}\psi = F_g(\psi)x_g(\psi) + G_g(\psi)q(\psi) \tag{4}$$

where ψ is central angle, and $q(\cdot)$ is a vector white Gaussian noise process. The output equation to form the disturbance terms is

$$u(\psi) = C_g(\psi)x_g(\psi) \tag{5}$$

The white noise terms, $q(\psi)$, are modeled to have correlation

$$\mathcal{E}[q(\psi)q^T(\psi+\Delta\psi)] = Q_g(\psi)\delta(\Delta\psi) \tag{6}$$

where $\delta(\cdot)$ is the Dirac delta function.

The spatial equation of Eq. (4) can be converted to a temporal equation when the trajectory is assumed to be a great circle one with monotonic ψ of less than 180 deg. For this case,

$$\mathrm{d}\psi/\mathrm{d}t = v/r \tag{7}$$

where v is horizontal velocity, and r is radius magnitude. With cosmetic changes in the functions,

$$\dot{x}_g(t) = F_g(t)x_g(t) + G_g(t)q(t) \tag{8}$$

An augmented state can now be defined as

$$x_a(t) = \left[\begin{array}{c} x(t) \\ \hdashline x_g(t) \end{array}\right] \tag{9}$$

The augmented state vector differential equation is

$$\dot{x}_a(t) = F_a(t)x_a(t) + G_a(t)q(t) \tag{10}$$

where

$$F_a(t) = \left[\begin{array}{c:c} F(t) & G(t)C_g(t) \\ \hdashline 0 & F_g(t) \end{array}\right] \tag{11}$$

and

$$G_a(t) = \left[\begin{array}{c} 0 \\ \hdashline G_g(t) \end{array}\right] \tag{12}$$

With this augmented form the system covariance integral becomes

$$P_a(t) = \int_{t_0}^{t}\int_{t_0}^{t} \Phi_a(t,p)G_a(p)Q_g(p)\delta(q-p) \times G_a^T(q)\Phi_a^T(t,q)\mathrm{d}p\mathrm{d}q \tag{13}$$

where

$$\dot{\Phi}_a(t,t_I) = F_a(t)\Phi_a(t,t_I) \tag{14a}$$

subject to

$$\Phi_a(t_I,t_I) = I \tag{14b}$$

and where initial INS covariances have been assumed to be 0. A more general formulation is possible which would consider, for example, initial INS misalignment errors due to vertical defletions.[5] The delta function allows a reduction to

$$P_a(t) = \int_{t_0}^{t} \Phi_a(t,p)G_a(p)Q_g(p)G_a^T(p)\Phi_a^T(t,p)\mathrm{d}p \tag{15}$$

Applying Leibnitz' rule now yields

$$\dot{P}_a(t) = F_a(t)P_a(t) + P_a(t)F^T(t) + G_a(t)Q_g(t)G_a^T(t) \tag{16}$$

This equation can be solved numerically by a number of powerful and efficient numerical integration techniques. This efficiency is the most persuasive argument for using linear state-space methods.

Obviously, the linear state-space covariance analysis restricts the gravity disturbance statistical model to a form compatible with linear state-space description. The restriction, while undesirable, is much less serious than the loss of trajectory generality, which accompanies the time-domain use of the Gauss-Markov model. Gravity disturbance is a spatial function and, hence, the statistical representation is a spatial process. The conversion from spatial to a temporal representation of the statistical process causes the resulting model to err when the trajectory varies from the great circle, constant-altitude form.

To understand why this conversion creates error, one need only consider the variable involved. The spatial model has central angle ψ as an independent variable. Central angle is not a scalar since, in general, the total central angle change is not equal to the sum of the central angle changes along mission segments. Time is a scalar, however, and the conversion through Eq. (7) to the time as the independent variable, means that the underlying relationship is mismodeled, in general. In the case of a closed-course trajectory, the gravity model error repeats. The correlation of errors must also repeat, but the time domain Markov model completely fails in this regard. A numerical example presented later demonstrates the seriousness of this mismodeling. For the special case of a great circle trajectory, the total central angle change is the sum of the segment changes as long as the nondecreasing central angle is less than 180 deg.

Another trajectory restriction is imposed by the original spatial Gauss-Markov model. This formulation does not treat changes in altitude, so the model is only valid for constant altitude trajectories as well. Equivalent statistics may be calculated from the original at other altitude levels[12]; however, combining these into a linear system format requires some improvisation.

In summary, the only trajectory for which the linear state-space covariance analysis is valid is a constant altitude, great circle case with a monotonically nondecreasing central angle of less than 180 deg. So, the considerable computation efficiency achieved through Eq. (16) has the severe trajectory limitations as a price. Although the trajectory restrictions of the Gauss-Markov model are understood, the extent of errors when the trajectory restrictions are violated is not known. The numerical example presented later explores this area.

II. The Covariance Integral Approach

Adequate models exist for trajectory, for navigation system, and for gravity disturbance statistics, but the two present methods of statistical analysis have serious disadvantages that limit their usefulness. An alternative method is needed, more flexible than linear state-space covariance analysis, and more efficient than Monte Carlo.

To develop an efficient yet flexible alternative analysis method, a line of development is suggested which is parallel to the linear state-space method, but which avoids the dependence on the Gauss-Markov statistical model. The crucial change-of-variables (from central angle ψ to time t) does not occur in this new "covariance integral" formulation; the spatial dependence of the correlations is retained. This approach does not exclude the Gauss-Markov gravity disturbance statistical model from use. But it does mean that the gravity disturbance correlations from Eq. (4) are used directly, and the temporal linear filter model of Eq. (8) is avoided entirely. As a consequence, the integral expression of the navigation error covariance will not contain a simplifying Dirac delta function, as in Eq. (13). The covariance integral becomes, then, the end object of the theoretical development. The final analysis will be simply a numerical approximation of this integral.

To generalize the approach leading to Eq. (13), the Gauss-Markov model of Eq. (4) is not assumed, and a covariance integral is produced using the INS error states of Eq. (3). The expectation operator applied to the outer product of $x(t)$, which is found by integrating Eq. (3), yields

$$P(t) = \int_{t_0}^{t} \int_{t_0}^{t} \Phi(t,p) G(p) Q(p,q) G^T(q) \Phi^T(t,q) \mathrm{d}p \mathrm{d}q \quad (17)$$

where Φ and G are from the INS error-state model of Eq. (3),

$$Q(p,q) = \mathcal{E}[u(p) u^T(q)] \quad (18)$$

and $x(t_0)$ is assumed to be zero.

This geodetic error correlation function, $Q(p,q)$, summarizes the statistical relations between gravity errors at time p with those at time q. Since $u(p)$ is, in reality, a function of the position $r(p)$ defined by the trajectory, it is more correct to express this function in terms of positions (i.e., $Q[r(p),r(q)]$). The trajectory model yields the required position history, $r(p)$ and $r(q)$, for evaluations of the correlation function model.

Applying Leibnitz' rule to Eq. (17) yields

$$\dot{P}(t) = F(t)P(t) + P(t)F^T(t) + G(t)D(t) + D^T(t)G^T(t) \quad (19a)$$

where

$$D(t) = \int_{t_0}^{t} Q(t,p) G^T(p) \Phi^T(t,p) \mathrm{d}p \quad (19b)$$

Since time appears as an argument of $Q(\cdot,\cdot)$, differentiation of Eq. (19b) will not yield a simple form, in general. Combining Eq. (19b) with the integral form of Eq. (19a) yields a pair of nested single integrals, which can be approximated, as an alternative to approximating Eq. (17).

III. Numerical Method

The objective, to provide a more flexible method than linear state-space covariance analysis, has been met by the above developments. The second objective, to provide a more efficient method than Monte Carlo analysis, depends on the numerical algorithm. Approximating the covariance integral is straightforward, but computational cost becomes prohibitive for realistic problems. The data logistics issue is critical in forming an economically feasible analysis method.

Two general lines can be followed to approximate the covariance integral. A direct approximation can be made by replacing the double integration of Eq. (17) with finite double summations. That is, using discrete notation,

$$P(n) = \sum_{i=0}^{n} \sum_{j=0}^{n} \Phi(t_n,t_i) G(t_i) Q(t_i,t_j) \times G^T(t_j) \Phi^T(t_n,t_j) \Delta t_i \Delta t_j s_n(i,j) \quad (20)$$

where $s_n(i,j)$ is the quadrature weighting associated with the integrand evaluation at $(p,q) \equiv (t_i,t_j)$, and Δt_i is the time increment associated with the t_i. The alternative Nested Integrals method is an approximation of Eqs. (19b) and (19a) in that order. The form of Eq. (19a) permits the application of efficient predictor-corrector numerical integration once Eq. (19b) is numerically approximated. Recursive algorithms were developed along both these theoretically equivalent lines.

For either the direct method or Nested Integrals, a discrete representation of the mission time (t_0,t_f) is required. To aid in this partitioning, one can view the integrals of Eq. (17) as a process applied to a signal, the integrand. With this perspective, the time steps should be small enough to give at least adequate representation of the integrand as specified by Shannon's sampling theorem. The frequency spectrum of the integrand is affected by all three analysis models. The trajectory affects both the Φ and Q matrices. The inertial navigation error propagation model, along with the design mission trajectory model, define the F matrix used in computing Φ. The gravity disturbance correlation model is embodied in Q, which has position arguments defined by the trajectory. The dynamics of the trajectory, the dynamics of the error propagation model, and the spatial spectrum of the gravity disturbance correlation model must all be considered in subdividing the time interval.

Once a proper time partition is defined, the integrand must be evaluated for every pair (t_i,t_j) such that both t_i and t_j are in the set $\{t_0,t_1,...,t_N\}$. This Φ-matrix data need is a critical computational issue, which must be addressed.

If n_s is the number of error states, the integral summation can also be reduced from n_s^2 terms to $\frac{1}{2}n_s(n_s+1)$ because $P(t)$ is symmetric. This symmetric property will be used to decrease the number of $\Phi(t_i,t_j)$-matrices. That is, the only $\Phi(t_i,t_j)$-matrices required are for (i,j) pairs such that $0\leq j\leq i\leq N$. Since $\Phi(t_i,t_j)=I$ when $i=j$, the true requirement is for (k,j) pairs such that $0\leq j<i\leq N$. For example, $\Phi(t_4,t_7)$ need not be calculated, whereas $\Phi(t_7,t_4)$ is required. The total number of Φ-matrices required is $\frac{1}{2}(N+1)(N+2)-(N+1)=\frac{1}{2}N(N+1)$. To demonstrate the magnitude of this data burden, consider a 500-point time partition and 9×9 Φ-matrices. The 125, 250 Φ-matrices would require 10, 145, 250 words of storage.

Clearly the storage and retrieval of these data must be carefully planned in order to make this analysis method economically feasible. A detailed account of the computer implementation is presented in Ref. 5; the overall approach can be summarized as four techniques. First, only incremental transition matrices, $\Phi(t_{i+1},t_i)$, were stored, and the semigroup property was used to reconstruct any $\Phi(t_{i+j},t_i)$ required. The summations were then ordered to bring in data in a reverse chronological order, thereby eliminating wasted effort in forming the Φ-matrices. Next, consistent with this order, data files were in reverse chronological order, and index sequential file organization was used for maximum efficiency of data recall. Finally, several sets of data were packed into each record to lower the number of data-fetch operations. The data could then be processed through the summations in a manner efficient in both computational and input-output time.

To compare the direct approximation to Nested Integrals, a simple test case was constructed for which a closed-form solution could be given. For this problem the simple Schuler loop model was used with exponentially correlated gravity deflections similar to Ref. 10. The two approaches gave approximately equal numerical accuracy, but Nested Integrals computational costs were one-third of the direct approximation costs. It was also shown that little accuracy is gained by smaller time divisions, once the Shannon sampling rate is attained in both the error propagation dynamics model and the correlation model. The Nested Integrals method was used in the remainder of this study.

The Nested Integrals algorithm consisted of a two-step solution. First, using Eq. (19b), $D(t_n)$ was calculated for $n=1,2,...,N$, using

$$D(t_n)=\sum_{i=0}^{n}Q[r(t_n),r(t_{n-i})]G^T(t_{n-i})\times\Phi^T(t_n,t_{n-i})\Delta t_{n-i}s_n(t_{n-i}) \tag{21}$$

where Δt_{n-i} is the time increment, and

$$s_n(t_j)=1 \qquad (1\leq j\leq n-1)$$
$$=\tfrac{1}{2} \qquad (j=0 \text{ or } n) \tag{22}$$

Note that the reverse chronological form of Eq. (21) means that $\Phi(t_n,t_j)$ can be formed using $\Phi(t_n,t_{j+1})$ from the previous integrand evaluation postmultiplied by $\Phi(t_{j+1},t_j)$ from data retrieval. With the sequence $\{D(t_1)D(t_2),...,D(t_N)\}$ calculated, the covariances $\{(P(t_0),...,P(t_N)\}$ can be found by solving Eq. (19a). For this, a predictor-corrector integration approximation was used with $D(t)$ values interpolated across the time intervals.

IV. Nested Integrals vs Linear State Space

The Nested Integral algorithm allows an analyst complete freedom to choose both a trajectory model and a gravity disturbance statistical model. By selecting a disturbance correlation model consistent with linear state-space methods, an interesting comparison can be made between Nested Integrals and the linear state-space method.

The three models must be specified for this study. The Widnall-Grundy[3] (pp. 26-27) navigation error propagation model was used with vertical channel damping simulated as suggested by Britting[2] [let $\kappa=3$ in Eq. (8-114)]. This nine-state error propagation model was coupled to an eight-state gravity disturbance model,[6] which evolved from the original Levine and Gelb work.[10] For study purposes, a 20 n.mi. correlation distance, and an 1800 (mgal)^2 radial disturbance variance were used as reasonable approximation to an empirical correlation function.[8] The constant altitude trajectory was varied to 1) demonstrate agreement on the great circle case, and 2) demonstrate the error of linear state-space method when the trajectory restriction is violated. For all cases, a constant 615 ft/s horizontal velocity was used, a constant altitude was assumed, and other variables (e.g., heading) were allowed to change consistent with the required trajectory geometry.

On the great circle case, both methods should give correct covariances. The results (Fig. 2) were virtually identical. The Nested Integrals solution required nearly three times the computational time of the linear state-space method, however.

To demonstrate the errors in the classical linear state-space approach, minor circle trajectories of varying radii were simulated. At a minor circle radius of one correlation distance (20 n.mi.), considerable error is observed (Fig. 3). For such a complex trajectory, the efficiency of the linear state-space method would be of little value since the results are clearly wrong.

With these studies, the Nested Integrals approximation of the covariance integrals has been verified against the linear state-space covariance analysis on the great circle case. The minor circle case, then, shows the error the linear state-space method is subject to when the great circle trajectory restriction is violated. The objective to create a more flexible analysis has been achieved.

V. Nested Integrals vs Monte Carlo

The second objective was to develop an analysis method more efficient than Monte Carlo. To investigate this aspect, the study performed by Chatfield et al.[9] of gravity-induced navigation errors on a hypothetical air-launched cruise missile has many features which prompted its use:

1) Complex trajectory. A high-g boost climb phase giving significant altitude and velocity changes compared to the previous studies.

2) Correlation model variation. The Tscherning-Rapp anomaly-degree-variance statistical model[7] was used.

3) Real world problem. The Monte Carlo study was not an academic study but was a part of an engineering trade-off on proposed new systems.

4) Independent verification. These published results[9] offer an independent check on the validity of Nested Integrals solutions.

In the Nested Integrals study, an attempt was made to match the Monte Carlo trajectory. The trajectory starts at 500 ft altitude, and 615 ft/s horizontal velocity. In the first 10 s of flight, the missile climbs to 80,000 ft and reaches a cruise speed of 3900 ft/s. A 1500 n.mi. flight was considered.

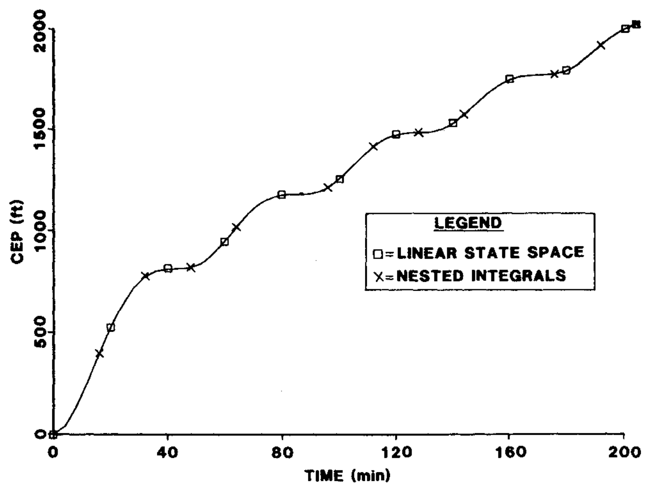

Fig. 2 Great circle case.

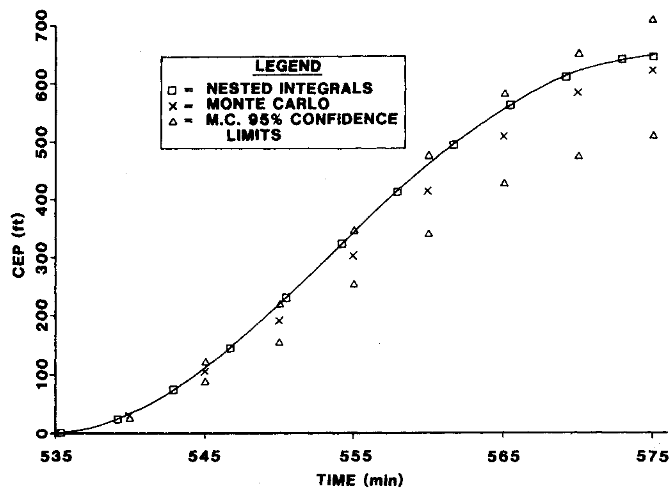

Fig. 4 Cruise missile case.

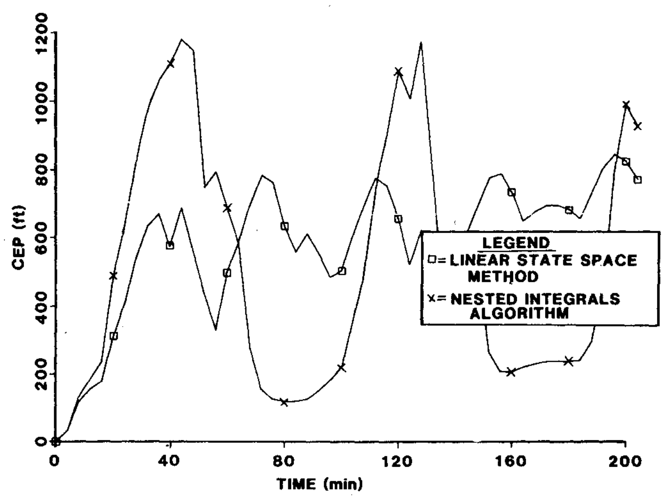

Fig. 3 Minor circle case.

Monte Carlo analysis used a whole-valued navigation model; the Nested Integrals method could not match this method exactly. The modified Widnall-Grundy error propagation model, which was used in the great circle case, was again employed.

The results from Nested Integrals analysis were very close to Monte Carlo results (Fig. 4), in spite of the trajectory and navigation system model differences. The computation times were not close, however. In fact, the Monte Carlo study took 30 times more computer central processor time (on the same system) than the Nested Integral study. The new analysis method, then, is indeed more efficient than Monte Carlo analysis, while delivering equivalent results on a complex mission.

VI. Kalman Filter Updates

The previous development has provided no means of incorporating the effects of measurement updates on the navigation error process. Since high-accuracy missions include such updates when possible, such an omission seriously limits the usefulness of this method.

The Nested Integrals approach can be modified rather simply to accommodate these updates. If the updates occur at $\{t_{n_i}\}$, a subsequence of the $\{t_n\}$ sequence of the Eq. (19b) integrand evaluations, the Kalman gain matrix $K(t_{n_i})$ is calculated for each such point. A discrete error propagation matrix can be formed by

$$\Phi(t_{n_i}^+, t_{n_i}^-) = I - K(t_{n_i})H(t_{n_i}) \tag{23}$$

where $H(\cdot)$ is the measurement matrix, the + superscript denotes "after update," and the − superscript denotes "before update." With this matrix, the covariance update is

$$P(t_{n_i}^+) = \Phi(t_{n_i}^+, t_{n_i}^-)P(t_{n_i}^-)\Phi^T(t_{n_i}^+, t_{n_i}^-) \tag{24}$$

Since this analysis treats gravity model errors alone, there is no measurement noise term in Eq. (24). $D(t)$ solution of Eq. (19b) must use

$$D(t_{n_i}^+) = D(t_{n_i}^-)\Phi^T(t_{n_i}^+, t_{n_i}^-) \tag{25}$$

at each of the measurement points. Of course, the error state transition matrix must also reflect these effects using the semigroup property

$$\Phi(t_{n_i}^+, t) = \Phi(t_{n_i}^+, t_{n_i}^-)\Phi(t_{n_i}^-, t) \tag{26}$$

Incorporation of Eqs. (24-26) into the numerical solution of Eqs. (19a) and (19b) gives the ability to account for the effects of Kalman filter updates on navigation errors induced by gravity model errors.

To verify the performance of the approach, position measurements were simulated along the previously studied great circle mission. Nested Integral results were again virtually identical to the linear state space covariance analysis results.

VII. Conclusions

A new statistical analysis technique has been developed and verified. The covariance integral approach gives more flexibility in both the trajectory model and the gravity disturbance statistical model. The Nested Integrals solution of the covariance integral provided equivalent results to Monte Carlo at a considerable savings in computational resources. The ability to incorporate effects of updates was provided and verified. Overall, the Nested Integrals method provides an efficient new method of evaluating the inertial navigation systems gravitation model in terms of system-level accuracy on realistic scenarios.

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